Journal of Algebra **214,** 652–667 (1999)
Article ID jabr.1998.7710, available online at http://www.idealibrary.com on IDEAL®

Doubles, Finiteness Properties of Groups, and Quadratic Isoperimetric Inequalities*

Martin R. Bridson

Communicated by Peter M. Neumann

Received September 16, 1997

We describe a doubling construction that gives many new examples of groups that satisfy a quadratic isoperimetric inequality. Using this construction, we prove that the presence of a quadratic isoperimetric inequality does not constrain the higher finiteness properties of a group (in contrast to the sub-quadratic case).

INTRODUCTION

This article concerns a connection between two topics in geometric group theory that have received a good deal of attention in recent years—finiteness properties and isoperimetric inequalities. We shall examine how finiteness properties and isoperimetric inequalities behave with respect to certain doubling operations on groups. The well-known groups of Stallings [St] and Bieri [Bi1] can be described in terms of such doubling operations, and using this description we shall show that these groups satisfy a quadratic isoperimetric inequality (Theorem B).

The class $\mathscr{IP}(2)$ of groups that satisfy a quadratic isoperimetric inequality is known to be extensive but is far from understood. $\mathscr{IP}(2)$ contains all finitely generated abelian groups and free groups, automatic groups, fundamental groups of compact non-positively curved spaces, $SL_n(\mathbb{Z})$ for $n \geq 4$, various nilpotent groups N, and those non-uniform lattices in rank 1 Lie groups that have these N as cusp groups. This list of examples is a

E-mail: bridson@maths.ox.ac.uk
* This work was supported by NSF Grant DMS-94011362 and an EPSRC Advanced Fellowship. I thank Robert Bieri and Chuck Miller for their helpful comments concerning this article.

0021-8693/99 $30.00

fair representation of the current state of knowledge but it is probably not a fair representation of the actual range of groups in $\mathcal{IP}(2)$. Theorem A, which is stated below, will allow us to construct many new examples.

We adopt the notation $\Delta_2(G; H)$ for the double of a group G along a subgroup H, i.e., the amalgamated free product of two copies of G with the two copies of H identified by the identity map. More generally, we write $\Delta_m(G; H)$ for the amalgamated free product of m copies of G along H. The Dehn function f_G of a group G describes the optimal isoperimetric inequality satisfied by G (see Section 1).

THEOREM A. *Let G_1 and G_2 be finitely presented groups with epimorphisms to $\mathbb{Z}$, let $\phi : G_1 \times G_2 \twoheadrightarrow \mathbb{Z}$ be the induced map, and let $K = \ker \phi$. For every $m \geq 2$, the group $\Delta_m(G_1 \times G_2; K)$ is finitely presented and its Dehn functions is $\simeq \max\{f_{G_1}, f_{G_2}\} + n^2$.*

This theorem is of most interest in the case where G_1 and G_2 both satisfy a quadratic isoperimetric inequality; such examples are given in Sections 2 and 3.

If a group satisfies a sub-quadratic isoperimetric inequality then it is hyperbolic in the sense of Gromov [Gr1] and hence enjoys many properties that seem unrelated to the complexity of the word problem—higher finiteness properties, solvable conjugacy problem, finitely many conjugacy classes of finite subgroups, etc. At present it is unknown whether the groups in $\mathcal{IP}(2)$ share any significant properties beyond the complexity of their word problem (although there is some evidence that they do, e.g., [P] versus [Br1]). The previously known examples of groups in $\mathcal{IP}(2)$ enjoy all of the common higher finiteness conditions listed in (3.1). However, using Theorem A we shall prove that none of these conditions is enjoyed by all of the groups in $\mathcal{IP}(2)$.

The first example of a finitely presented group that is not of type F_3 was discovered by John Stallings [St]. His example was re-interpreted by Robert Bieri [Bi1], who constructed the first sequence of groups[1] $(SB_n)_{n \in \mathbb{N}}$ such that SB_n is of type F_{n-1} but not of type F_n. (A concise presentation of SB_n is given below in Corollary 2.9.) Steve Gersten proved that each of the groups SB_n satisfies a polynomial isoperimetric inequality [Ge]. He obtained a quintic bound for SB_3 and this was later sharpened to cubic by Baumslag *et al.* [BBMS].

In the following theorem, if $m = 2$ then $K_n \cong SB_n$.

THEOREM B. *Let L be a free group of rank $m \geq 2$ and let $L^{(n)}$ denote its n-fold direct product. Let $\phi : L \twoheadrightarrow \mathbb{Z}$ be an epimorphism and let K_n be the*

[1] The notation SB_n is in honour of Stallings and Bieri.

kernel of the induced map $L^{(n)} \twoheadrightarrow \mathbb{Z}$. *For every* $n \geq 1$,

(1) $K_{n+1} \cong \Delta_m(L^{(n)}; K_n)$;

(2) K_n *is of type* F_{n-1} *but* $H_n(K_n, \mathbb{Z})$ *is not finitely generated*;

(3) K_n *satisfies a quadratic isoperimetric inequality* (*and this is optimal*).

COROLLARY C. *Let* D *be a direct product of free groups and let* $\psi : D \twoheadrightarrow \mathbb{Z}$ *be an epimorphism. If the kernel of* ψ *is finitely presented, then it satisfies a quadratic isoperimetric inequality.*

In a subsequent article [Br2], Theorem A and certain of the ideas underlying it (as exposed in Section 3) will be used to address the question of which finitely presented groups are fundamental groups of compact non-positively curved spaces.

1. ISOPERIMETRIC INEQUALITIES AND THEOREM A

1.1. *Preliminaries*. Isoperimetric inequalities measure the complexity of the word problem in finitely presented groups by giving a bound on the number of relators that one must apply in order to determine whether or not a word in the generators represents the identity. More precisely, given a finite presentation $\langle \mathscr{A} | \mathscr{S} \rangle$ of the group Γ, the number of relators that one must apply in order to show that a word w in the free group $F(\mathscr{A})$ represents $1 \in \Gamma$ is denoted $\mathrm{Area}(w)$;

$$\mathrm{Area}(w) := \min\left\{ N \,\middle|\, w = \prod_{i=1}^{N} x_i^{-1} s_i x_i \text{ (freely)} \right.$$
$$\left. \text{for some } x_i \in F(\mathscr{A}),\ s_i \in \mathscr{S}^{\pm 1} \right\},$$

and the *Dehn function* of Γ is defined by

$$f_\Gamma(n) := \max\{\mathrm{Area}(w) \mid w =_\Gamma 1, |w| \leq n\}.$$

If there exists a constant α such that $f_\Gamma(n) \leq \alpha n^2$ for all $n \in \mathbb{N}$, then one says that Γ satisfies a *quadratic isoperimetric inequality*.

Two functions $f, g : \mathbb{N} \to \mathbb{N}$ are said to be $\simeq$ equivalent if $f \preccurlyeq g$ and $g \preccurlyeq f$, where $f \preccurlyeq g$ means that there exists a constant $C > 0$ such that $f(n) \leq Cg(Cn) + Cn + C$ for all $n \in \mathbb{N}$. Up to $\simeq$ equivalence, f_Γ is independent of the choice of finite presentation [A1].

The suggestive terms "area" and "isoperimetric" are indicative of the fact that one can interpret $\mathrm{Area}(w)$ as the least number of 2-cells in any van Kampen diagram for w (see [LS]), and using this combinatorial notion of area one can interpret isoperimetric inequalities for groups in the language of isoperimetric inequalities for Riemannian manifolds.

It is straightforward to show that for any infinite finitely presented groups G_1 and G_2 one has $f_{G_1 \times G_2} \simeq \max\{f_{G_1}, f_{G_2}\} + n^2$. It is equally straightforward to show that if $H \subset \Gamma$ is a retract then $f_H \preccurlyeq f_\Gamma$. ($H \subset \Gamma$ is called a *retract* if there exists a homomorphism $\Gamma \to H$ that restricts to the identity on H.)

Proof of Theorem A. For any pair of groups $H \subset \Gamma$, there is an obvious retraction of $\Delta_m(\Gamma; H)$ onto Γ, and from the remarks in the previous paragraph it follows that the formula given in Theorem A for the Dehn function of $\Delta_m(G_1 \times G_2; K)$ is at least a lower bound. A more subtle argument is required in order to show that it is also an upper bound.

1.2. *Notation.* We have a map $\phi : G_1 \times G_2 \twoheadrightarrow \mathbb{Z}$ that restricts to $\phi_i : G_i \twoheadrightarrow \mathbb{Z}$ for $i = 1, 2$. Let $K = \ker \phi$. As generating sets for G_1 and G_2 we choose $\mathscr{B}_1 \cup \{\alpha\}$ and $\mathscr{B}_2 \cup \{\beta\}$, where $\phi_1(\alpha) = \phi_2(\beta)$ is a generator of $\mathbb{Z}$ and $\mathscr{B}_i \subset \ker \phi_i$. Note that $\alpha\beta^{-1} \in K$. We shall work with a fixed finite presentation

$$G_1 \times G_2 = \langle \mathscr{B}, \alpha, \beta \mid \mathscr{R} \rangle,$$

where $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$ and $\mathscr{R}$ includes all of the relations $[x_1, x_2]$ with $x_1 \in \mathscr{B}_1 \cup \{\alpha\}$ and $x_2 \in \mathscr{B}_2 \cup \{\beta\}$.

1.3. LEMMA. *$K = \ker \phi$ is generated by $\mathscr{B} \cup \{\alpha\beta^{-1}\}$.*

Proof. Each $k \in K$ can be expressed in $G_1 \times G_2$ as $k = k_1 k_2$, where k_i is a word in the chosen generators for G_i. Let e_1 and e_2 be the sum of the exponents of all occurrences of α and β, respectively. Notice that $e_1 = -e_2$. Let $\hat{k}_1$ be the word obtained from k_1 by replacing each α^ε with $(\alpha\beta^{-1})^\varepsilon$ and let $\hat{k}_2$ be the word obtained from k_2 by replacing each $\beta^{-\varepsilon}$ with $(\alpha\beta^{-1})^\varepsilon$. Using the commutators that we included in $\mathscr{R}$, we see that $k = \hat{k}_1(\beta^{e_1}\alpha^{e_2})\hat{k}_2 = \hat{k}_1(\alpha\beta^{-1})^{e_2}\hat{k}_2$. ∎

1.4. COROLLARY. *The following is a finite presentation for $\Delta_m(G_1 \times G_2; K)$,*

$$\langle \mathscr{B}, \theta, \alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m \mid S, \mathscr{R}_1, \dots, \mathscr{R}_m, \theta = \alpha_i \beta_i^{-1}, i = 1, \dots, m \rangle,$$

where $\mathscr{R}_i$ denotes the set of relations obtained from $\mathscr{R}$ by replacing each occurrence of α and β with α_i and β_i, and where S consists of all the commutators $[\theta, \alpha_i]$ and $[\theta, \beta_i]$. (The relations S follow from the others that are stated but it is convenient to include them anyway.)

For the remainder of this section we work with the presentation described in (1.4).

1.5. LEMMA. $\underline{a} := \{\alpha_1, \ldots, \alpha_m\}$ *freely generates a free subgroup of* $\Delta_m(G_1 \times G_2; K)$.

Proof. The quotient of $\Delta_m(G_1 \times G_2; K)$ by K is a free group of rank m generated by the images of the α_i. ∎

Let w_0 be a word of length n in the generators of $\Delta_m(G_1 \times G_2; K)$ given in (1.4) and suppose that $w_0 = 1$.

We Must Estimate Area(w_0)

We shall give a purely algebraic description of the moves by which w_0 can be reduced to the empty word. Readers who prefer diagrammatic arguments should have no difficulty translating each move into the language of van Kampen diagrams.

An Outline of the Strategy. We apply relations so as to replace w_0 by a word w of length at most $9n$ that has the normal form displayed in $(*)$ below. Then, by removing pinches (Step 3) and consolidating (Step 4) we replace w by a word of length $|w|$ that involves only the generators $\mathscr{B}$ and θ. We then replace each occurrence of θ by $\alpha_1 \beta_1^{-1}$ and view the resulting word as a word in the generators of $G_1 \times G_2$. As such it can be reduced to the empty word by applying at most $f_{G_1 \times G_2}(|w|)$ relators from $\mathscr{R}_1$. Moreover, by estimating the number of relations applied at each stage, we find that it is the cost of this final reduction that gives the dominant term, which is $\simeq f_{G_1 \times G_2}(n)$.

We introduce one more piece of notation. Recall that $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2$. Let

$$X_i := \mathscr{B}_i \cup \{\theta\}.$$

Notice that among the relators of our presentation (1.4) we have all of the commutators $[x_1, \alpha_i]$ with $x_1 \in X_1$ and $[x_2, \beta_i]$ with $x_2 \in X_2$. Thus, for example, we may commute any occurrence of α_i past any word U in the generators X_1 at the cost of applying exactly $|U|$ relators.

Step 1. *Removing all* $\beta_{i \neq 1}$. We temporarily remove from w_0 all occurrences of θ, replacing them with $\alpha_1 \beta_1^{-1}$. Then we replace each occurrence of $\beta_{i \neq 1}$ by $\beta_1 \alpha_1^{-1} \alpha_i$. These operations increase the length of w_0 by at most a factor of 3, and the number of relators that have to be applied is less than $2n$.

Gathering terms (but not applying any relators) we obtain $w_1 = u_1 v_1' \ldots u_r v_r'$, where the u_i are words in the generators $\mathscr{B} \cup \{\beta_1, \alpha_1\}$ and the v_i are words in the generators $\underline{a} - \{\alpha_1\}$.

Step 2. *A normal form*. Each u_j defines an element of $G_1 \times G_2$. If $\phi(u_j)$ is m_j times the chosen generator of $\mathbb{Z}$, then $u_j \alpha_1^{-m_j} \in K$. Now, $|m_j| \le |u_j|$, so by applying at most $(2|u_j|)^2$ commutator relations from $\mathscr{R}_1$, we may replace $u_j \alpha_1^{-m_j}$ by $k_1 k_2$, where k_1 is a word in the generators $\mathscr{B}_1 \cup \{\alpha_1\}$ and k_2 is a word in the generators $\mathscr{B}_2 \cup \{\beta_1\}$. Note that $|k_1| + |k_2| \le 2|u_j|$. Proceeding exactly as in the proof of Lemma 1.3, by applying less than $(4|u_j|)^2$ additional commutator relations we can replace $k_1 k_2$ by $\hat{k}_1(\alpha\beta^{-1})^{e_j}\hat{k}_2$. Finally, we replace $(\alpha\beta^{-1})^{e_j}$ by θ^{e_j}, and write $u_{j,1} = \hat{k}_1 \theta^{e_j}$ and $u_{j,2} = \hat{k}_2$. Thus $u_j = u_{j,1} u_{j,2} \alpha_1^{m_j}$.

By performing the above sequence of moves on each u_j in w_1 and writing $v_j = \alpha_1^{m_j} v'_j$ we obtain the word

$$w = \prod_{j=1}^{r} (u_{j,1} u_{j,2}) v'_j, \qquad (*)$$

where the $u_{j,1}$ are all words in the generators X_1, the $u_{j,2}$ are all words in the generators X_2, and the v_j are non-trivial elements of the free group on $\underline{a}$. We shall regard words as being in *normal form* if they are expressed as a product $(*)$ satisfying these conditions.

The length of w is at most $3|w_1|$, and in passing from w to w_1 we have used less than $21|w_1|^2$ relators. (Here we have implicitly used the fact that $\Sigma|u_j|^2 \le (\Sigma|u_j|)^2$.)

Step 3. *Removing Pinches*. If we were to delete all of the subwords $u_{j,i}$ from w in $(*)$ we would obtain a word in the free group on $\underline{a}$ that was *freely* equal to the identity. It follows that w must contain a *pinch*, i.e., a subword of the form $\alpha_i^{\varepsilon} u_{j,1} u_{j,2} \alpha_i^{-\varepsilon}$. For convenience we assume that $\varepsilon = 1$.

We have the free equality

$$\alpha_i u_{j,1} u_{j,2} \alpha_i^{-1} = \left(\alpha_i \beta_i^{-1}\right)\left[\beta_i u_{j,1} \beta_i^{-1}\right]\left(\beta_i \alpha_i^{-1}\right)\left[\alpha_i u_{j,2} \alpha_i^{-1}\right].$$

We apply the relator $\theta = \alpha_i \beta_i^{-1}$ to replace the subwords in round brackets, and we simplify the expressions in square brackets using the remark immediately preceding Step 1. Thus at a cost of applying $|u_{j,1}| + |u_{j,2}| + 2$ relators we can replace the subword $\alpha_i u_{j,1} u_{j,2} \alpha_i^{-1}$ by $(\theta u_{j,1})(\theta^{-1} u_{j,2})$. The two factors in this last expression are words over the alphabets X_1 and X_2, respectively; we relabel them $u'_{j,1}$ and $u'_{j,2}$, respectively. Thus at the cost of applying less than $|w|$ relators we have replaced a pinch $\alpha_i u_{j,1} u_{j,2} \alpha_i^{-1}$ in w by a word of the same length that has the form $u'_{j,1} u'_{j,2}$.

Step 4. *Consolidation.* If we remove pinches as in Step 3 then the redacted version of w may no longer be in normal form $(*)$. For example, if we removed the visible pinch from $(u_{1,1} u_{1,2} \alpha_1 u_{2,1} u_{2,2} \alpha_1^{-1} \ldots)$, then we would be left with a word that began $(u_{1,1}u_{1,2}u'_{2,1}u'_{2,2}\dots)$. We must rewrite this last expression as $(U_1U_2\dots)$ with U_1 and U_2 words over the alphabets X_1 and X_2, respectively. In order to do so we commute $u'_{2,1}$ (to the left) past $u_{1,2}$; this can be done at the cost of applying at most $|u_{1,2}|\cdot|u'_{2,1}|$ relators because in (1.4) we included all of the commutators $[x_1, x_2]$ with $x_i \in X_i$.

Step 5. *The Final Step*. Given w as in $(*)$ we remove a pinch from it. If the redacted word remains in normal form $(*)$, then we remove another pinch, if not then we consolidate the subword $\dots u_{j,1}u_{j,2}u_{j+1,1}u_{j+1,2}\dots$ that violates the normal form condition, as in Step 4. Notice that neither of these operations alters the length of w (provided that we do not freely reduce the resulting word). We continue to remove pinches and consolidate until all occurrences of the letters $\underline{a}$ have been removed. We then have a word of length $|w|$ in the generators $\mathscr{B}\cup\{\theta\}$. We replace each occurrence of θ by $\alpha_1\,\beta_1^{-1}$ and view the resulting word as a word in the generators of $G_1\times G_2$. As such it can be reduced to the empty word at the cost of applying at most $f_{G_1\times G_2}(|w|)$ relators from $\mathscr{R}_1$.

1.6. *Counting the Cost*. The cost of Step 1 (i.e., the total number of relators applied) was at most $2n$. The total cost of Step 2 was less than $21|w_1|^2$, which is no more than $189n^2$. In Step 3, the removal of each pinch can be achieved at the cost of applying at most $|w|$ relators. Since each pinch reduces the total number of occurrences of letters from $\underline{a}$, the total cost of all pinches is less than $|w|^2 \le 81n^2$.

The cost of the consolidation described in Step 4 was $|u_{j,2}|\cdot|u'_{j+1,1}|$. Calculating the total cost of all consolidation is complicated somewhat by the fact that the subwords $u_{j,i}$ (and their indices) change as we alter w by removing pinches and consolidating. But they change in a very simple way: during pinching some new symbols $\theta^{\pm1}$ are introduced and during consolidation $u_{j,i}$ and $u_{j+1,i}$ may get concatenated; at every stage the sum of the lengths of the $u_{j,i}$ remains bounded by $|w|$. The cost of consolidation comes from the need to commute $u_{j+1,1}$ to the left past $u_{j,2}$. The key point to observe is that once we have commuted a given symbol past another we never have to do so again (with the same symbols). Therefore the total cost of all of consolidations is bounded by $|w|^2$.

The cost of the Step 5 is bounded by $|w| + f_{G_1\times G_2}(|w|)$. Thus

$$\mathrm{Area}(w_0) \le 11n + 351n^2 + f_{G_1\times G_2}(9n).$$

The dominant term here is $f_{G_1\times G_2}(9n)$, which is $\simeq \max\{f_{G_1}(n), f_{G_2}(n)\} + n^2$. This completes the proof of Theorem A. ∎

Isoperimetric Inequalities for Doubles

Let $H \subset G$ be a pair of finitely generated groups. H is said to be quasi-isometrically embedded in G if for some (hence any) choice of word metrics d_H and d_G, there is a constant C such that $\frac{1}{C}d_G(h, h') \leq d_H(h, h') \leq Cd_G(h, h')$ for all $h, h' \in H$. The proof of (1.3) shows that the inclusion $K \hookrightarrow G_1 \times G_2$ is a quasi-isometric embedding. Using this observation, one can give a much easier proof of Theorem A in the case where the Dehn functions of G_1 or G_2 are such that $nf_{G_i}(n) \simeq f_{G_i}(n)$ (which happens if the f_{G_i} are exponential, for example). This result is a special case of results in Chapter III.Γ of [BH] bounding the Dehn function of $\Delta_m(G; H)$ in terms of f_G and the distortion of H in G (in the sense of [Gr2]).

1.7. Proposition. *Let $m \geq 2$, let G be a finitely presented group, and let $H \subset G$ be a finitely generated subgroup that is quasi-isometrically embedded. Then the Dehn function of $f_{\Delta_m(G;\, H)}$ satisfies*

$$f_G(n) \preccurlyeq f_{\Delta_m(G;\, H)}(n) \preccurlyeq nf_G(n).$$

The lower bound in the above proposition is sharp because G is a retract of $\Delta_m(G; H)$. The following example shows that one cannot improve the upper bound without further hypotheses.

1.8. Example. Let $G = L \times L$, where L is a free group of rank 2. Choose generators a, b for the first factor and s, t for the second factor. Let H be the subgroup generated by $\{as, bs\}$. It is easy to see that H is quasi-isometrically embedded in G (it is even a retract), so (1.7) gives a cubic upper bound on the Dehn function of $\Delta_2(G; H) = G_1 *_H G_2$. One can prove that this bound is sharp by using a direction diagrammatic argument to show that $\mathrm{Area}[a_1^n b_1^{-n}, (t_1t_2)^n] \simeq n^3$ (see [BH]).

2. FINITENESS PROPERTIES AND THEOREM B

We recall the various ways of measuring the higher finiteness properties of groups. (See [Bi1, Bro, Se, and Wa] for more details.)

2.1. Definition. Let $n \in \mathbb{N}$. A group Γ is said to be of type F_n if $\Gamma = \pi_1 X$, where X is a CW complex with a contractible universal cover and finite n-skeleton.

A group Γ is said to be of type FP_n (resp. FL_n) if $\mathbb{Z}$, regarded as a trivial module over the group ring $\mathbb{Z}\Gamma$, admits a projective (resp. free) resolution in which the first $n + 1$ resolving modules are finitely generated.

The condition F_0 is vacuous. Γ is of type F_1 if and only if it is finitely generated. Γ is of type F_2 if and only if it is finitely presentable. If we write $H_n(G)$ to denote the nth homology group of G with integral coefficients, we have,

$$F_n \Rightarrow FL_n \Rightarrow FP_n \Rightarrow H_n(G) \text{ f.g.}$$

The study of finiteness properties has been pursued vigorously by the school of Robert Bieri and others (see, for example, [BS, BNS, Bi3], and references therein). It has recently been invigorated with an infusion of new geometric ideas by Bestvina and Brady [BB] (see also [MMV]). Their work began with a new perspective on the following examples of Stallings and Bieri.

2.2. *The Groups of Stallings and Bieri.* Let Λ be a free group of rank two. Let $\Lambda^{(n)}$ denote the direct product of n copies of Λ, and let $h_n : \Lambda^{(n)} \to \mathbb{Z}$ be a homomorphism that restricts to an epimorphism on each factor. Let $SB_n = \ker h_n$. Inspired by an example of John Stallings [St], Robert Bieri [Bi] showed that SB_n is of type FP_{n-1} but not type FP_n. Theorem B provides a short proof of this fact (in the spirit of the original) and at the same time shows that these groups satisfy a quadratic isoperimetric inequality.

THEOREM B. *Let L be a free group of rank $m \geq 2$ and let $L^{(n)}$ denote its n-fold direct product. Let $\phi : L \twoheadrightarrow \mathbb{Z}$ be an epimorphism and let K_n be the kernel of the induced map $L^{(n)} \twoheadrightarrow \mathbb{Z}$. For every $n \geq 1$,*

(1) $K_{n+1} \cong \Delta_m(L^{(n)}; K_n)$;

(2) *K_n is of type F_{n-1} but $H_n(K_n, \mathbb{Z})$ is not finitely generated*;

(3) *K_n satisfies a quadratic isoperimetric inequality* (*and this is optimal*).

Before turning to the proof of this result, we explain how it implies the following consequence that we stated in the Introduction.

COROLLARY C. *Let P be a direct product of free groups and let $\psi : P \twoheadrightarrow \mathbb{Z}$ be an epimorphism. If the kernel of ψ is finitely presented, then it satisfies a quadratic isoperimetric inequality.*

Proof. Since the kernel is assumed to have a finite generating set, there is no loss of generality in assuming that P is the product of finitely many free groups each of which is finitely generated. We can decompose P as a direct product $P_1 \times P_2$, where the restriction of ψ to P_1 is trivial and its restriction to each of the direct summands of P_2 is non-trivial. It suffices to show that if the kernel of $\psi|_{P_2}$ is finitely presented then it satisfies a quadratic isoperimetric inequality.

By passing to a subgroup of finite index, we may assume that ψ restricts to an epimorphism on each direct summand of P_2. Note that if one of the summands of P_2 is free of rank 1, then the kernel of $\psi|_{P_2}$ is just a product of free groups and we are done. (Given any group G and a map $h: G \times \langle t \rangle \to \mathbb{Z}$ such that $h(t) = -1$, there is an isomorphism from G to $\ker h$ given by $g \mapsto gt^{h(g)}$.)

Thus we may assume that the direct summands of P_2 all have rank at least 2. Let m be the greatest rank among the summands and let Λ be a free group of rank m. Let $\phi: \Lambda \twoheadrightarrow \mathbb{Z}$ be an epimorphism.

Given any free group L and an epimorphism $f: L \twoheadrightarrow \mathbb{Z}$, one can choose a basis $\mathscr{B} \cup \{y\}$ for L such that $f(y)$ generates $\mathbb{Z}$ and $f(b) = 0$ for all $b \in \mathscr{B}$. We pick a basis for Λ in line with this observation, casting ϕ in the role of f, and we pick a basis for each summand L_i of P_2 by casting $\psi|_{L_i}$ in the role of f. By identifying our chosen basis of L_i with an appropriate subset of the chosen basis of the ith summand of $\Lambda^{(n)}$, we can identify $P_2 = L_1 \times \cdots \times L_n$ with a subgroup of $\Lambda^{(n)}$ in such a way that the natural retraction from $\Lambda^{(n)}$ onto P_2 restricts to a retraction from the kernel of the map $\Lambda^{(n)} \to \mathbb{Z}$ induced by ϕ onto $\ker \phi|_{P_2}$. Similarly, there is a subgroup of P_2 isomorphic to SB_n onto which $\ker \psi|_{P_2}$ retracts.

If $\ker \psi|_{P_2}$ is finitely presented then its retract SB_n must also be finitely presented and hence $n \geq 3$. But then, by Theorem B(3), the kernel of $\Lambda^{(n)} \to \mathbb{Z}$ satisfies a quadratic isoperimetric inequality, and hence so does its retract $\ker \psi|_{P_2}$. ∎

The first of the following lemmas provides a tool for showing that groups are *not* of type F_n, the second provides a tool for showing that groups *are* of type F_n, and the third will be used as the base step of the induction by which we shall deduce Part (2) of the above theorem from part (1).

2.3. LEMMA. *Let $\Gamma = A *_C B$. If A and B are of type F_n but $H_{n-1}(C)$ is not finitely generated, then $H_n(\Gamma)$ is not finitely generated.*

Proof. Consider the Mayer–Vietoris sequence for $\Gamma = A *_C B$:

$$\begin{aligned} \cdots &\to H_n(A) \oplus H_n(B) \to H_n(\Gamma) \to H_{n-1}(C) \\ &\to H_{n-1}(A) \oplus H_{n-1}(B) \to \ \ldots . \end{aligned}$$

2.4. COROLLARY. *Let A and C be finitely generated groups, let $m \geq 2$ be an integer, and let $D_m = \Delta_m(A; C)$. If A is of type F_n but $H_{n-1}(C)$ is not finitely generated, then $H_n(D_m)$ is not finitely generated.*

Proof. D_m retracts onto $D_2 = A *_C A$. Thus $H_n(D_2)$ is a direct summand of $H_n(D_m)$, and we can apply the lemma. ∎

2.5. LEMMA. *If A is of type F_n and C is of type F_{n-1} then $\Delta_m(A;C)$ is of type F_n.*

Proof. Let X_A be an Eilenberg–Maclane complex for A that has a finite n-skeleton. Let X_C be an Eilenberg–Maclane space for C that has a finite $(n-1)$-skeleton. Let $f: X_C \to X_A$ be a cellular map that induces the inclusion $C \hookrightarrow A$. One obtains an Eilenberg–Maclane complex for $\Delta_m(A;C)$ by taking $(m-1)$ disjoint copies of the double mapping cylinder for f and identifying them along the X_A at the initial end of each copy:

$$X = \frac{X_A \cup \coprod_{i=2}^{m}\big((X_A, i) \cup (X_C \times [0,1], i)\big)}{\substack{(f(x), i) \sim (x, 1, i) \text{ and } (x, 0, i) \sim f(x) \\ \text{for all } x \in X_C \text{ and } i = 2, \ldots, m}}.$$

The open n-cells in X are of two kinds: there are the open n-cells in the images of X_A, and there are the products $e \times (0,1)$ where e is an open $(n-1)$-cell in X_C. Thus X has only finitely many n-cells. ∎

LEMMA 2.6. *Let L be a free group and let $N \subset L$ be a non-trivial normal subgroup. If L/N is infinite then $H_1(N)$ is not finitely generated.*

Proof. L is the fundamental group of a graph X with a single vertex. Let $\hat{X} = \tilde{X}/N$. L/N acts freely and transitively on the vertices of $\hat{X}$, so if L/N is infinite then $\hat{X}$ contains infinitely many disjoint loops and therefore $N = \pi_1 \hat{X}$ is freely generated by an infinite set. ∎

Proof of Theorem B. Part (3) follows from part (1), Theorem A, and the (easy) fact that $L^{(n)}$ satisfies a quadratic isoperimetric inequality. In order to prove that (1) implies (2), we argue by induction on n. The case $n = 1$ is covered by (2.6). In the inductive step we need the observation that the Cartesian product of n graphs of genus m is an Eilenberg–Maclane space for $L^{(n)}$, and hence $L^{(n)}$ is of type F_r for every $r \in \mathbb{N}$.

Suppose that $K_n \subset L^{(n)}$ is of type F_{n-1} but that $H_n(K_n)$ is not finitely generated. According to (2.4), the $(n+1)$st integral homology of $\Delta_m(L^{(n)}; K_n)$ is not finitely generated. And according to (2.5), $\Delta_m(L^{(n)}; K_n)$ is of type F_n. This completes the induction.

In order to prove part (1), we need two additional lemmas.

2.7. LEMMA. *If $N \subset G$ is normal then $\Delta_m(G; N)$ embeds in $(*_{i=1}^{m} G/N) \times G$.*

Proof. Let $\phi: \Delta_m(G; N) \to *_{i=1}^{m} G/N$ be the quotient by N, and let $\psi: \Delta_m(G; N) \to G$ be the natural retraction onto the first copy of G. Because $\ker \phi \cap \ker \psi = \{1\}$, $\Phi := (\phi, \psi)$ is injective. ∎

2.8. LEMMA. *Let G be a group. Let $h: G \to \langle \tau \rangle$ be an epimorphism to an infinite cyclic group and let $N = \ker h$. Choose $a \in G$ with $h(a) = \tau$ and let $\bar{a}$ denote the image of a in G/N. Let $\hat{h}: (*_{i=1}^{m} G/N) \times G \to \langle \tau \rangle$ be the map that is given on each of the factors G/N by $\hat{h}(\bar{a}) = -\tau$ and is such that $\hat{h}|_G = h$. Let Φ be as in (2.7).*

Then $\ker \hat{h} = \operatorname{im} \Phi$.

Proof. $\Delta_m(G; N)$ is generated by N and $a_1, \ldots, a_m$, where $\Phi(a_i) = (\bar{a}_i, a)$ and $\bar{a}_i$ generates the ith free factor of $*_{i=1}^{m} G/N$. For each $g \in N$ we have $\Phi(g) = (1, g)$, and hence $\hat{h}\Phi(g) = h(g) = 0$. For each a_i we have $\hat{h}\Phi(a) = \hat{h}(\bar{a}_i, a) = -\tau + \tau$. Thus $\operatorname{im} \Phi \subset \ker \hat{h}$.

Now suppose $(u, v) \in \ker \hat{h}$. Write u as a reduced word $\bar{a}_{i(1)}^{s_1}, \ldots, \bar{a}_{i(r)}^{s_r}$ and let

$$\gamma = \left(a_{i(1)}^{s_1} \cdots a_{i(r)}^{s_r}\right) a_1^{-h(v)} v.$$

Obviously, $a_1^{-h(v)} v$ lies in $N = \ker h$, so the first coordinate of $\Phi(\gamma)$ is u. On the other hand, because $-\hat{h}(u) = \hat{h}(v) = h(v)$, we have $\sum s_i = h(v)$, and therefore the second coordinate of $\Phi(\gamma)$ is v. ∎

We return to the proof of Theorem B(1). From Lemma 2.7 we get an embedding $\Delta_m(L^{(n)}; K_n) \hookrightarrow (*_{i=1}^{m} \mathbb{Z}) \times L^{(n)} \cong L^{(n+1)}$, and Lemma 2.8 shows that (modulo the natural identifications) the image of this embedding is K_{n+1}. ∎

The first part of Theorem B provides a simple presentation of K_n. We make this explicit in the case $m = 2$, where K_n is the Stallings–Bieri group SB_n.

2.9. PROPOSITION (A Presentation for SB_n). *If $n \geq 2$ then there is a presentation for SB_{n+1} with $3n$ generators $x_1, \ldots, x_n, y_1, y_1', \ldots, y_n, y_n'$ and relations*

$$y_1^{-1} y_j = y_1'^{-1} y_j', \qquad [x_i, x_j] = [x_i, y_j] = [x_i, y_j'] = [y_i, y_j] = [y_i', y_j'] = 1, \quad 1 \leq i < j \leq n.$$

Proof. Let L_2 be a free group with basis $\{x, y\}$ and let $x_i, y_i, i = 1, \ldots, n$ be the obvious generators for $L_2^{(n)}$. The group SB_n is the kernel of the map $\Psi : L_2^{(n)} \twoheadrightarrow \mathbb{Z}$ that sends each x_i to the identity and each y_i to a fixed generator of $\mathbb{Z}$.

The presentation displayed above describes $\Delta_2(L_2^{(n)}; H)$, where H is the subgroup generated by $\{x_1, \ldots, x_n, y_1^{-1} y_2, \ldots, y_1^{-1} y_n\}$. According to Theorem B(1), in order to prove the present proposition we must show that $H = \ker \Psi$. It is clear that $H \subset \ker \Psi$. Conversely, suppose that $\gamma = \gamma_1 \cdots \gamma_n \in \ker \Psi$, where γ_i is a word in the generators x_i and y_i.

Each γ_i is equal in the free group on $\{x_i, y_i\}$ to a word of the form

$$\left(\prod_{k=1}^{N_i} y_i^{p_{i,k}} x_i^{\varepsilon_{i,k}} y_i^{-p_{i,k}}\right) y_i^{q_i}.$$

For example, $y_i^2 x_i y_i x_i^2 = (y_i^2 x_i y_i^{-2})(y_i^3 x_i^2 y_i^{-3}) y_i^3$.

If $i \geq 2$ then $(y_1 y_i) x_i (y_1 y_i)^{-1} = y_i x_i y_i^{-1}$, and we also have $(y_1 y_2) x_1 (y_1 y_2)^{-1} = y_1 x_1 y_1^{-1}$. Hence all elements of the form $y_i^{p_{i,k}} x_i^{\varepsilon_{i,k}} y_i^{-p_{i,k}}$ are contained in H. Thus in order to show that $\ker \Psi \subset H$, it suffices to consider elements of the form $\gamma = y_1^{q_1} \cdots \gamma_n^{q_n} \in \ker \Psi$. But in this case we have $\sum_i q_i = 0$, and hence

$$\gamma = \prod_{i=1}^{n} \left(y_1^{-1} y_i\right)^{q_i},$$

which is in H. ∎

3. CLOSING REMARKS

3.1. EXAMPLES. There are lots of finitely presented groups that satisfy a quadratic isoperimetric inequality and have infinite abelianization. Such examples serve as building blocks to which we may apply Theorem A to obtain a large and diverse subclass of $\mathcal{IP}(2)$. In particular, since we place no finiteness conditions on the kernels of the maps $G_i \twoheadrightarrow \mathbb{Z}$ considered, the groups $\Delta_m(G_1 \times G_2; K) \in \mathcal{IP}(2)$ yielded by Theorem A will enjoy a variety of finiteness conditions. Moreover, an analysis of centralizers shows that even when these groups have a finite Eilenberg−Maclane space they are generally not bicombable (see [Br2]).

The fundamental groups of hyperbolic 3-manifolds that fibre over the circle [T] provide a rich class of building blocks G_i, as do the free-by-cyclic groups studied by Bestvina and Feighn [BF] and the fundamental groups of many cube complexes [BB]. All infinite Coxeter groups satisfy a quadratic isoperimetric inequality (see [Mou, AB]) and have a subgroup of finite index that maps onto $\mathbb{Z}$ (see [CLR, Go]). Many small cancellation groups also have infinite abelianization [LS].

Further candidates for the G_i can be obtained by applying the Rips construction [Ri] to any group Q with infinite abelianization. (Rips' construction assigns to each finite group representation a short exact sequence $1 \to N \to \Gamma \to Q \to 1$, where Γ is a hyperbolic group, N is finitely generated, and Q is the group given by the original presentation.)

3.2. *Semidirect Product Decompositions*. We adopt the notation $K \rtimes_{(\psi)} Q$ for a semidirect product in which Q has a preferred finite generating set each element of which acts on K by the automorphism $\psi \in \mathrm{Aut}(K)$.

For $i = 1, 2$, let N_i be the kernel of the map $G_i \twoheadrightarrow \mathbb{Z}$ considered in Theorem A and let K be the kernel of the induced map $G_1 \times G_2 \twoheadrightarrow \mathbb{Z}$. Let $\hat{G}_i = gp\{N_i, \alpha\beta^{-1}\}$ (notation of (1.3)). Note that $\hat{G}_i \cong G_i$. The subgroups N_1 and N_2 are normal in K and

$$K = N_1 \rtimes \hat{G}_2 = N_2 \rtimes \hat{G}_1.$$

By choosing a splitting of $1 \to K \to G_1 \times G_2 \to \mathbb{Z} \to 1$ we get $G_1 \times G_2 = K \rtimes_\psi \mathbb{Z}$. It follows that

$$\Delta_m(G_1 \times G_2; K) \cong K \rtimes_{(\psi)} L_m,$$

where L_m is free of rank m. This decomposition and the isomorphism in part (1) of Theorem B can be viewed in the following more general context.

A Generalization of Theorem B(1)

Given a group K_0 and $\psi \in \mathrm{Aut}(K_0)$, let $G_0 = K_0 \rtimes_\psi \mathbb{Z}$ and define sequences of groups G_n and K_n inductively by $G_n = G_0 \times L_m^{(n)}$ and $K_n = K_0 \rtimes_{(\psi)} L_m^{(n)}$, where L_m is a free group of rank $m \geq 2$ and $L_m^{(n)}$ is its n-fold Cartesian product.

3.3. PROPOSITION. (1) *There is a natural embedding $K_n \hookrightarrow G_n$, the image of which is a normal subgroup with quotient $\mathbb{Z}$.*

(2) $\Delta_m(G_n; K_n) \cong K_{n+1}$.

Proof. Consider $G_0 = K_0 \rtimes_\psi \langle t \rangle$. Fix a basis $s_1, \ldots, s_m$ for L_m and take the induced generators $\{s_{1,1}, \ldots, s_{m,n}\}$ for $L_m^{(n)}$. Note that G_n can be expressed as a semidirect product

$$G_n = K_0 \rtimes_{(\psi)} \left(\hat{L}_m^{(n)} \times \langle t \rangle\right),$$

where $\hat{L}_m^{(n)}$ is the subgroup with generating set $\{s_{i,j}t \mid 1 \leq i \leq m, 1 \leq j \leq n\}$.

There is a natural isomorphism from $K_n = K_0 \rtimes_{(\psi)} L_m^{(n)}$ to G_n, sending $K_0 \subset K_n$ to $K_0 \subset G_n$ by the identity and sending the chosen generators of $L_m^{(n)}$ to the generators $s_{i,j}t$ of $\hat{L}_m^{(n)}$. The image of this monomorphism is the kernel of the retraction $G \to \langle t \rangle$ implicit in the semidirect product decomposition displayed above. Moreover, since the conjugation action of t on $K_0 \subset G_n$ is by ψ and its action on $\hat{L}_m^{(n)}$ is trivial, if we identify $K_n = K_0 \rtimes_{(\psi)} L_m^{(n)}$ with its image in G_n, then the induced action of t on K_n is by $(\psi, 1)$. Therefore

$$\Delta_m(G_n; K_n) \cong \left(K_0 \rtimes_{(\psi)} L_m^{(n)}\right) \rtimes_{(\psi,1)} L_m \cong K_0 \rtimes_{(\psi)} L_m^{(n+1)}.$$

∎

The automorphism ψ in the definition of G_n does not influence the Dehn function of K_n:

3.4. COROLLARY. *If G_0 is finitely presented, then for every $n \geq 2$ the Dehn function of K_n is $f_{K_n}(i) \simeq \max\{f_{G_0}(i), i^2\}$.*

Proof. K_n is the kernel of the map $G_0 \times L_m^{(n)} \to \mathbb{Z}$ whose restriction to G_0 is the quotient map by K and whose restriction to $L_m^{(n)}$ sends each of the preferred generators to the generator of $\mathbb{Z}$.

Theorem A tells us that $\Delta_m(G_0 \times L_m^{(n)}; K_n)$ has Dehn function $\max\{f_{G_0}(i), i^2\}$, and part (2) of the preceding proposition completes the proof. ∎

REFERENCES

[A1] J. M. Alonso, Inégalités isopérimétriques et quasi-isométries, *C.R. Acad. Sci. Paris Sér. I* **311** (1990), 761–764.

[AB] J. M. Alonso and M. R. Bridson, Semihyperbolic groups, *Proc. London Math. Soc.* (3) **70** (1995), 56–114.

[BBMS] G. Baumslag, M. R. Bridson, C. F. Miller III, and H. B. Short, Finitely presented subgroups of automatic groups and their isoperimetric inequalities, *J. London Math. Soc.* (2) **56** (1997), 292–304.

[BB] M. Bestvina and N. Brady, Morse theory and finiteness properties of groups, *Invent. Math.* **129** (1997), 445–470.

[BF] M. Bestvina and M. Feighn, A combination theorem for negatively curved groups, *J. Differential Geom.* **35** (1992), 85–101.

[BH] M. R. Bridson and A. Haefliger, "Metric Spaces of Non-positive Curvature," Grundlehren Math. Wiss., Springer-Verlag, Berlin/Heidelberg/New York, 1999.

[Bi1] R. Bieri, Homological dimension of discrete groups, Queen Mary Lecture Notes, 1976.

[Bi2] R. Bieri, Normal subgroups in duality groups and in groups of cohomological dimension 2, *J. Pure Appl. Algebra* **7** (1976), 32–52.

[Bi3] R. Bieri, Finiteness length and connectivity length for groups, *in* "Group Theory Down Under" (J. Cossey, C. F. Miller III, W. D. Neumann, and M. Shapiro, Eds.), de Gruyter, 1998.

[BNS] R. Bieri, W. D. Neumann, and R. Strebel, A geometric invariant of discrete groups, *Invent. Math.* **90** (1987), 451–477.

[Br1] M. R. Bridson, Asymptotic cones and polynomial isoperimetric inequalities, *Topology* **38** (1999), 543–554.

[Br2] M. R. Bridson, On the subgroups of semihyperbolic groups, preprint, University of Oxford, 1998.

[BS] R. Bieri and R. Strebel, Geometric invariants for discrete groups, preprint of book, Frankfurt, 1992.

[Bro] K. S. Brown, "Cohomology of Groups," Springer-Verlag, New York, 1985.

[CLR] D. Cooper, D. Long, and A. Reid, Infinite Coxeter groups are virtually indicable, *Proc. Edinburgh Math. Soc. 2* **41** (1998), 303–313.

[Ge] S. M. Gersten, Finiteness properties of asynchronously automatic groups, *in* "Geometric Group Theory" (R. Charney, W. Neumann, and M. Shapiro, Eds.), de Gruyter, Berlin, 1996.

[Go] C. Gonciulea, Infinite Coxeter groups virtually surject onto $\mathbb{Z}$, *Comment Math. Helv.* **72** (1997), 257–265.

[Gr1] M. Gromov, Hyperbolic groups, *in* "Essays on Group Theory" (S. M. Gersten, Ed.), MSRI Publications, Vol. 8, Springer-Verlag, New York/Berlin, 1987.

[Gr2] M. Gromov, Asymptotic invariants of infinite groups, *in* "Geometric Group Theory, Vol. 2" (G. Niblo and M. Roller, Eds.), London Math. Soc. Lecture Note Ser., Vol. 181, Cambridge Univ. Press, Cambridge, UK, 1993.

[LS] R. C. Lyndon and P. E. Schupp, "Combinatorial Group Theory," Springer-Verlag, Berlin, 1977.

[MMV] J. Meier, H. Meinert, and L. van Wyck, Higher generation subgroup sets and the Σ-invariants of graph groups, *Comment. Math. Helv.* **73** (1998), 22–44.

[Mou] G. Moussong, "Hyperbolic Coxeter Groups," Ph.D. Thesis, The Ohio State University, 1988.

[P] P. Papasoglou, Asymptotic cones and the quadratic isoperimetric inequality, *J. Differential Geom.* **44** (1996), 789–806.

[Ri] E. Rips, Subgroups of small cancellation groups, *Bull. London Math. Soc.* **14** (1982), 45–47.

[Se] J.-P. Serre, Cohomologie des groupes discrets, *Ann. Math. Stud.* **70** (1971), 77–169.

[St] J. R. Stallings, A finitely presented group whose 3-dimensional integral homology is not finitely generated, *Amer. J. Math.* **85** (1963), 541–543.

[T] W. P. Thurston, Three dimensional manifolds, Kleinian groups and hyperbolic geometry, *Bull. Amer. Math. Soc.* **6** (1982), 357–381.

[Wa] C. T. C. Wall, Finiteness conditions for CW complexes, *Ann. of Math.* **81** (1965), 56–69.